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A double-scale investigation of the asymptotic structure of rolled-up vortex sheets

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A double scale technique is used to determine the asymptotic behaviour of a rolled-up vortex sheet. The technique relies on a process of averaging out the saw-tooth-like behaviour of the flow variables, which generates a continuous solution having the structure of a vortex filament. The fine-scale behaviour of the flow is described and includes concentrated vorticity on the sheet. Application to the conical vortex sheet allows the solution of Mangler & Weber (1967) to be rederived. A further application, to Kaden's problem, is worked out and the results are in complete agreement with Moore's asymptotic formulae for the shape of the spiral.

1. Introduction

We start from the simple idea that a vortex sheet is an infinitely narrow region carrying infinite vorticity, a concept which may be given a precise meaning with the aid of distribution theory; the vorticity of the sheet at a given point is then a Dirac delta function times $\mathbf{n} \wedge [\mathbf{u}]$, where $\mathbf{n}$ is the unit normal to the sheet and $[\mathbf{u}]$ the discontinuity in the velocity across it. Accordingly we speak of $\mathbf{n} \wedge [\mathbf{u}]$ as the vorticity of the sheet. If many vortex sheets each of which carries a weak vorticity are embedded within a region of flow, then, in some sense, even if the flow between the sheets is irrotational it may, in an approximate manner, be regarded as one without any sheet and with continuously distributed vorticity. This approach may be useful in some circumstances even if it is not physically meaningful. But there are circumstances where this scheme of many sheets with weak vorticity actually occurs in the flow under investigation. This happens to be the case for the core of a rolled-up vortex sheet and it would be highly desirable to devise a mathematical technique which would allow the actual flow with a vortex sheet to be modelled by an equivalent (to some approximation) one with continuously distributed vorticity. This is particularly clear when one deals with numerical computations of rolled-up vortex sheets like those in Chorin & Bernard (1973) or Moore (1974) for the rolling-up of a trailing vortex sheet or that in Smith (1968) for the leading-edge vortex sheet of a flat delta wing at incidence in incompressible flow, which was described earlier by Roy (1957) and was replaced by concentrated vorticity in the preliminary attempt of Brown & Michael (1954) and in Rehbach (1975) for more complex leading-edge configurations.

Kaden (1931) has obtained an asymptotic representation of the core of the semi-infinite sheet emerging from the potential flow around a flat edge which is instantaneously removed, and this provides a model for the rolling-up of a trailing vortex sheet as described in Moore & Saffman (1973). Although it was most illuminating and essentially correct, the work of Kaden did not have the mathematical status which is now common in many rational, if not mathematically rigorous, asymptotic expansion theories. Recently Moore (1975) succeeded in achieving this goal for the Kaden type of result and found two terms in the asymptotic expansion of the equation of the spiral, which revealed a quite interesting effect, namely a departure from circular symmetry in the angle of pitch of the spiral in the first correction to Kaden's leading term $\theta \propto \gamma t r^{-\frac{2}{3}}$. Apart from the physically meaningless improvement of raising it to the status of a mathematically demonstrable theorem, there is one point in Moore's work which leaves room for a physically meaningful improvement. His theory is in fact devoted to obtaining an asymptotic representation of the equation of the spiral and some work has to be done in order to convert this into an asymptotic representation for the flow itself. As an application of the general theory to be developed in this paper we rederive Moore's expansion, adding a few further terms, and obtain, without any more calculation, the asymptotic representation of the velocity field. Mangler & Weber (1967) devised an inviscid theory which gives an asymptotic representation of the flow in the core of a rolled-up vortex sheet which is applicable to the delta leading-edge core and to Kaden's core. To be specific, let us consider the first application. The sheet is assumed to be conical and is found to spiral in a nearly circular way. For convenience we use conical similarity and restrict ourselves to what happens on the unit sphere. The equation of the spiral is derived to leading order in the distance to the focus of the spiral, while an asymptotic representation of the flow variables is given as the first two terms of what looks like an asymptotic expansion. The basic parameter of the expansion is, again, the distance (on the unit sphere with our convention) to the focus of the spiral. In fact, it is a co-ordinate-like expansion. The first term of this expansion is identical to the corresponding one in the solution for axially symmetric, rotational, inviscid, incompressible conical flow derived by Hall (1961). This means that, according to what was said above, the vorticity, which is concentrated on the sheet, with the form of a Dirac delta function in the exact solution, has been smeared out and appears as distributed vorticity. With the second term of the expansion we recover the discontinuous structure of the flow, and the two terms together provide an irrotational solution between the turns of the rolled sheet. The distributed vorticities associated with the first and second terms respectively cancel each other out. The reason why this is possible is that the second term has a built-in length scale in the direction normal to the sheet which is an order of magnitude lower than that of the leading term, so that vorticities of the two terms are of the same order.

Through inspection of the formulae derived by Mangler & Weber, it is apparent that their solution has a double scale structure. Although Mangler & Weber did not take advantage of this, it occurred to the present authors that Mangler & Weber's solution could best be derived by applying the technique of multiple scaling. It turned out that this view was correct and that, with a small amount of algebra, the expansion could be carried one step further. In fact, this higher-order correction to Mangler & Weber's solution is not the whole story. The point is that the correction which is derived in such a way does not exhibit the departure from circular symmetry of the spiral which should occur in the model if it is to be of some value for the numerical computation of the delta-wing problem. Indeed, the correction we give is only part of the full correction and is given for the purpose of illustration. As with any local asymptotic expansion, this one contains arbitrary constants or functions which can be computed only by matching with another, non-local, asymptotic expansion. We observe that at least two small parameters may be built into the problem of a rolled-up sheet: one is the slenderness parameter, which for the leading-edge conical sheet is the distance to the focus of the spiral, while the other is the reciprocal of the number of turns or the distance between turns, which we may call the closeness parameter. For the conical leading-edge vortex sheet it turns out that the second small parameter is of the order of the square of the first if we adopt the useful convention that logarithms are of order one. It is through the slenderness parameter that the core expansion is influenced by the exterior solution and thus has to take into account the departure from circular symmetry. For the leading-edge core, the very fact that closeness is slenderness squared means that the departure from circular symmetry should occur at the order after the one computed by Mangler & Weber. It was to satisfy our curiosity that the part of this correction corresponding to closeness has been computed, and we have included it in the paper. Thanks to the insistence of one referee, who was not satisfied by our argument that ellipticity of the spiral is related to the slenderness expansion and that it is worth while to have a separate process in order to deal with the closeness expansion, we have worked out Kaden's problem, which allows a clear understanding of the roles of the two expansions.

In the next two sections, we explain the main idea on which the expansion with respect to closeness relies, without any slenderness assumption, and delay to §§ 4 and 5 application of the technique to the leading-edge core and to Kaden's rolled-up vortex sheet respectively.

2. The double scale structure

We consider time-dependent, incompressible, irrotational flow with very many, closely spaced, vortex sheets, each carrying a weak vorticity. We work throughout with non-dimensional quantities and use t for time, $\mathbf{x}$ for vector position, $\mathbf{u}$ for velocity and p for pressure, the density being unity. We introduce a function $\chi(t, \mathbf{x})$ such that the overall equation of the many sheets is

$$\chi(t, \mathbf{x}) = (2k+1)\pi, \quad k = \ldots -2, -1, 0, 1, 2, \ldots, \tag{1}$$

and state the closeness assumption as

$$|\partial\chi/\partial t|, |\nabla\chi| \gg 1, \tag{2}$$

where ∇ stands for the gradient operator. A word of explanation about (1) may be useful. The whole sheet is in fact given by χ = constant, but it happens that, when the sheet is rolled-up around a line, the function $\chi(t, \mathbf{x})$ is multiple valued and this explains why the constant takes several values. In local cylindrical co-ordinates (r, ϕ, σ) the function χ has the structure $\chi = \phi + F(t, r, \sigma)$ and the sequence of values on the right-hand side of (1) is related to the multiple valuedness of ϕ. Now we come back to the problem and state the set of basic equations (not independent) that we use throughout:

$$\nabla . \mathbf{u} = 0, \tag{3}$$

$$\partial u/\partial t + (\mathbf{u} . \nabla)\,\mathbf{u} + \nabla p = 0, \tag{4}$$

$$\nabla \wedge \mathbf{u} = 0, \tag{5}$$

$$\partial\chi/\partial t + \mathbf{u} . \nabla\chi = 0 \quad \text{on both sides of each sheet,} \tag{6}$$

$$[p] = \nabla\chi . [\mathbf{u}] = 0 \quad \text{across each sheet,} \tag{7}$$

where $[f]$ stands for the discontinuity in f across the sheet counted from lower to higher values of χ.

In order to incorporate the closeness assumption in the model, we use the very popular technique of multiple scaling and averaging. More specifically, we set

$$\mathbf{u}(t, \mathbf{x}) = \mathbf{u}^*(t, \mathbf{x}; \chi(t, \mathbf{x})) \quad p(t, \mathbf{x}) = p^*(t, \mathbf{x}; \chi(t, \mathbf{x})) \tag{8}$$

and try to build up a solution with $\mathbf{u}^*$ and p^* functions of t, $\mathbf{x}$ and χ, considered as independent variables. Substituting (8) into (3)–(7) we get

$$\nabla\chi . \partial\mathbf{u}^*/\partial\chi + \nabla . \mathbf{u}^* = 0, \tag{9}$$

$$\left(\frac{\partial\chi}{\partial t} + \mathbf{u}^* . \nabla\chi\right)\frac{\partial\mathbf{u}^*}{\partial\chi} + \frac{\partial p^*}{\partial\chi}\nabla\chi + \frac{\partial\mathbf{u}^*}{\partial t} + (\mathbf{u}^* . \nabla)\,\mathbf{u}^* + \nabla p^* = 0, \tag{10}$$

$$\nabla\chi \wedge \partial\mathbf{u}^*/\partial\chi + \nabla \wedge \mathbf{u}^* = 0, \tag{11}$$

$$\partial\chi/\partial t + \mathbf{u}^* . \nabla\chi = 0, \quad \chi = (2k+1)\,\pi, \tag{12}$$

$$[p^*] = \nabla\chi . [\mathbf{u}^*] = 0 \quad \text{across} \quad \chi = (2k+1)\,\pi. \tag{13}$$

It is now clear how the closeness assumption can be used to obtain an approximation. Before entering into the formalities, we observe that, as in any multiple-scale technique, we shall encounter at each step a set of conditions that are to be enforced in order to eliminate secular terms. We may achieve this once and for all by averaging. Let us stress the fact that the way in which $\mathbf{u}^*$ and p^* depend on χ, on account of assumption (2), which states that χ varies rapidly across the space between two consecutive sheets, takes care of the main variation of $\mathbf{u}^*$ and p^* from one sheet to the next. Now, on $\chi = (2k+1)\,\pi$, $\mathbf{u}^*$ but not p^* suffers a discontinuity. For one type of solution that we consider here, each discontinuity cancels out the corresponding variation between the two consecutive sheets at least to leading order. This means that $\mathbf{u}^*$ has a saw-tooth-like appearance, as a function of χ, and the way to enforce the absence of secular terms is to demand that $\mathbf{u}^*$ and p^* be periodic as functions of χ. Then for any scalar or vector function of χ, we may conveniently set

$$\left.\begin{aligned} f^*(\chi) &= \overline{f^*} + \tilde{f}^*(\chi), \\ \overline{f^*} &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} f^*(\chi)\,d\chi, \quad \int_{-\pi}^{+\pi} \tilde{f}^*(\chi)\,d\chi = 0. \end{aligned}\right\} \tag{14}$$

For later convenience we refer to the process of going from any f^* to $\overline{f^*}$ as the averaging process and call $\tilde{f}^*$ the fluctuation in f.

Now we apply the averaging process to (9)–(11), taking due account of (12) and (13). From periodicity of $\mathbf{u}^*$ and p^* we get the set of averaged equations

$$\nabla . \overline{u^*} = 0, \tag{15}$$

$$\frac{\partial \overline{\mathbf{u}^*}}{\partial t} + \overline{(\mathbf{u}^* . \nabla)\,\mathbf{u}^*} + \nabla \overline{p^*} = -\overline{(\nabla . \mathbf{u}^*)\,\mathbf{u}^*}. \tag{16}$$

The way to obtain (15) is obvious when due account is taken of (7). Concerning (16) we observe that, thanks to (7) again, the term $(\partial p^*/\partial \chi)\,\nabla\chi$ averages to zero, while when averaging $(\partial\chi/\partial t + \mathbf{u}^* . \nabla\chi)\,\partial\mathbf{u}^*/\partial\chi$ we obtain, through one integration by parts, two terms, one of which is zero from (6) while the other is the average of $[(\partial\mathbf{u}^*/\partial\chi) . \nabla\chi]\,\mathbf{u}^*$, which is precisely $\overline{(\nabla . \mathbf{u}^*)\,\mathbf{u}^*}$, from (9).

We observe that these equations have some analogy with the usual equations of incompressible flow, the main difference being that $\overline{(\mathbf{u}^* . \nabla)\,\mathbf{u}^*}$ *is not* $(\overline{\mathbf{u}^*} . \nabla)\,\overline{\mathbf{u}^*}$, at least before any approximation. The averaging of (11) requires some care because this is not an equation in conservation form, but from $\mathbf{u}^*|_{\chi=-\pi}^{\chi=\pi} = -2\pi[\mathbf{u}^*]$ we get the following averaged form of the equation of cancellation of vorticity:

$$2\pi\nabla \wedge \overline{\mathbf{u}^*} = -\nabla\chi \wedge [\mathbf{u}^*], \quad \text{giving} \quad \nabla\chi . (\nabla \wedge \overline{\mathbf{u}^*}) = 0. \tag{17}$$

Now, it is obvious that if we could, through some approximation, replace $\overline{(\mathbf{u}^* . \nabla)\,\mathbf{u}^*}$ in (16) by $(\overline{\mathbf{u}^*} . \nabla)\,\overline{\mathbf{u}^*}$ we should have achieved the goal that we mentioned at the beginning of this paper. Equation (17) is quite illuminating in this respect. Let us call $(\overline{p^*}, \overline{\mathbf{u}^*})$ the model flow; we see that the vorticity of this model flow is exactly the result of continuously redistributing the Dirac-type vorticity which is concentrated on the sheet for the exact flow. We see also that the vorticity in the model flow is of order one as the result of the high value of $|\nabla\chi|$ and the weak vorticity of the sheet, namely $\mathbf{n} \wedge [\mathbf{u}^*]$, and we refer to our comment about the cancellation of vorticity for the first two terms of Mangler & Weber's solution.

3. Formalities

We intend to take advantage of (2) in order to solve (9)–(13), approximately, through an expansion process. We assume that, when applied to the flow variables $\partial/\partial t$ and ∇ are of order s^{-1}, where s is the so-called slenderness parameter. In the application to the leading-edge core s is found to be $O(\theta|\log\theta|)$ if θ is the angular distance to the focus of the spiral (on the unit sphere). We assume further that

$|\partial\chi/\partial t|$ and $|\nabla\chi|$ are or order $c^{-1}s^{-1}$, where c is the closeness parameter, which is supposed to be small. Let us consider (11); we rewrite it as

$$\nabla\chi \wedge \partial \mathbf{u}^*/\partial\chi + cs\nabla\wedge\mathbf{u}^* = 0, \tag{11a}$$

and then expand according to

$$\left.\begin{aligned} \mathbf{u}^* &= \mathbf{u}^{*(0)} + c\mathbf{u}^{*(1)} + c^2\mathbf{u}^{*(2)} + \dots = \mathbf{u}_0^* + \mathbf{u}_1^* + u_2^* + \dots, \\ p^* &= p^{*(0)} + cp^{*(1)} + c^2p^{*(2)} + \dots = p_0^* + p_1^* + p_2^* + \dots, \\ \chi &= \chi^{(0)} + c\chi^{(1)} + c^2\chi^{(2)} + \dots = \chi_0 + \chi_1 + \chi_2 + \dots. \end{aligned}\right\} \tag{18}$$

Instead of introducing the small parameter c into the equations we leave them as they stand and solve them iteratively in the obvious manner. As a matter of fact, in the application, the expansion with respect to c is a co-ordinate expansion and its structure is obvious from the results.

Substituting (18) into (9)–(13) we find easily

$$\partial\mathbf{u}_0^*/\partial\chi = \partial p_0^*/\partial\chi = 0, \quad \text{giving} \quad \overline{\mathbf{u}_0^*} = \mathbf{u}_0^* \quad \overline{p_0^*} = p_0^*, \tag{19}$$

and from (15) and (16) we see that, to the leading approximation, the model flow satisfies the continuity and Euler equations:

$$\left.\begin{aligned} &\nabla.\mathbf{u}_0^* = 0, \\ &\partial\mathbf{u}_0^*/\partial t + (\mathbf{u}_0^*.\nabla)\mathbf{u}_0^* + \nabla p_0^* = 0. \end{aligned}\right\} \tag{20}$$

On the other hand, from (12) we find

$$\partial\chi_0/\partial t + \mathbf{u}_0^*.\nabla\chi_0 = 0. \tag{21}$$

Proceeding to the second approximation, we find

$$\nabla\chi_0.\partial\mathbf{u}_1^*/\partial\chi = 0, \tag{22}$$

$$\nabla\chi_0 \wedge \partial\mathbf{u}_1^*/\partial\chi = -\nabla\wedge\mathbf{u}_0^*, \tag{23}$$

from which

$$\mathbf{u}_1^* = \overline{\mathbf{u}_1^*} + \frac{\nabla\chi_0\wedge\boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2} Y_1(\chi), \quad \boldsymbol{\omega}_0^* = \nabla\wedge\mathbf{u}_0^*, \tag{24}$$

$$p_1^* = \overline{p_1^*}, \tag{25}$$

where $Y_1(\chi)$ is defined by

$$Y_1(\chi) = \chi \quad \text{for} \quad |\chi| < \pi, \quad Y_1(\chi) \quad \text{periodic with period } 2\pi. \tag{26}$$

We must stress that (22) and (23) are soluble with respect to $\partial\mathbf{u}_1^*/\partial\chi$ if and only if

$$\nabla\chi_0.\boldsymbol{\omega}_0^* = 0. \tag{27}$$

From (15) and (16) we obtain

$$\nabla.\overline{\mathbf{u}_1^*} = 0, \tag{28}$$

$$\overline{\partial\mathbf{u}_1^*}/\partial t + (\mathbf{u}_0^*.\nabla)\overline{\mathbf{u}_1^*} + (\overline{\mathbf{u}_1^*}.\nabla)\mathbf{u}_0^* + \nabla\overline{p_1^*} = 0, \tag{29}$$

while from (12) we get

$$\partial\chi_1/\partial t + \mathbf{u}_0^*.\nabla\chi_1 + \overline{\mathbf{u}_1^*}.\nabla\chi_0 = 0. \tag{30}$$

We now examine the third approximation in order to exhibit the dynamical condition on the sheet. From (9)–(11) we find, taking (28) and (29) into account

$$\nabla\chi_0 \cdot \frac{\partial \mathbf{u}_2^*}{\partial \chi} + \nabla\chi_1 \cdot \frac{\partial \mathbf{u}_1^*}{\partial \chi} + \chi\nabla \cdot \frac{\partial \mathbf{u}_1^*}{\partial \chi} = 0, \tag{31a}$$

$$\nabla\chi_0 \wedge \frac{\partial \mathbf{u}_2^*}{\partial \chi} + \nabla\chi_1 \wedge \frac{\partial \mathbf{u}_1^*}{\partial \chi} + \chi\nabla \wedge \frac{\partial \mathbf{u}_1^*}{\partial \chi} + \nabla \wedge \overline{\mathbf{u}_1^*} = 0, \tag{31b}$$

$$\nabla\chi_0 \frac{\partial p_2^*}{\partial \chi} = -\chi\left\{\left(\frac{\partial}{\partial t} + \mathbf{u}_0^* \cdot \nabla\right)\frac{\partial \mathbf{u}_1^*}{\partial \chi} + \left(\frac{\partial \mathbf{u}_1^*}{\partial \chi} \cdot \nabla\right)\mathbf{u}_0^*\right\}, \tag{31c}$$

and from this, provided that

$$\nabla\chi_1 \cdot \boldsymbol{\omega}_0^* + \nabla\chi_0 \cdot (\nabla \wedge \overline{\mathbf{u}_1^*}) = 0 \tag{32}$$

holds, we get

$$\mathbf{u}_2^* = \overline{\mathbf{u}_2^*} + |\nabla\chi_0|^{-2}\left\{\nabla\chi_0 \wedge \left[\nabla\chi_1 \wedge \frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2} + \nabla \wedge \overline{\mathbf{u}_1^*}\right] - \left(\nabla\chi_1 \cdot \frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2}\right)\nabla\chi_0\right\} Y_1(\chi)$$
$$+ |\nabla\chi_0|^{-2}\left\{\nabla\chi_0 \wedge \left[\nabla \wedge \frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2}\right] - \left(\nabla \cdot \frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2}\right)\nabla\chi_0\right\} Y_2(\chi), \tag{33a}$$

$$p_2^* = \overline{p_2^*} - \frac{Y_2(\chi)}{|\nabla\chi_0|^2}\left\{\left(\frac{\partial}{\partial t} + \mathbf{u}_0^* \cdot \nabla\right)\frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2} + \left(\frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2} \cdot \nabla\right)\mathbf{u}_0^*\right\} \cdot \nabla\chi_0, \tag{33b}$$

where $$Y_2(\chi) = \tfrac{1}{2}(\chi^2 - \tfrac{1}{3}\pi^2) \quad \text{for} \quad |\chi| < \pi, \quad Y_2(\chi)\ 2\pi\text{-periodic}. \tag{34}$$

We observe that (31 *c*) is soluble for $\partial p_2^*/\partial \chi$ if and only if

$$\nabla\chi_0 \wedge \left\{\left(\frac{\partial}{\partial t} + \mathbf{u}_0^* \cdot \nabla\right)\frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2} + \left(\frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2} \cdot \nabla\right)\mathbf{u}_0^*\right\} = 0. \tag{35}$$

We may interpret this relation as follows. It is known that, along any vortex sheet with unit normal $\mathbf{n}$, the following dynamical condition holds:

$$\mathbf{n} \wedge \left\{\frac{D_m[\mathbf{u}]}{Dt} + ([\mathbf{u}] \cdot \nabla)\mathbf{u}_m\right\} = 0, \quad \nabla\chi = |\nabla\chi|\mathbf{n}, \tag{36}$$

with $$\left.\begin{aligned} D_m/Dt &= \partial/\partial t + \mathbf{u}_m \cdot \nabla; \\ \mathbf{u}_m &= \tfrac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-), \quad [\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-, \end{aligned}\right\} \tag{37}$$

where $\mathbf{u}^+$ and $\mathbf{u}^-$ are the values of $\mathbf{u}$ on either side of the sheet. The first approximation to (36) here reads

$$\mathbf{n}_0 \wedge \left\{\left(\frac{\partial}{\partial t} + \mathbf{u}_0^* \cdot \nabla\right)\frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2} + \left(\frac{\nabla\chi_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla\chi_0|^2} \cdot \nabla\right)\mathbf{u}_0^*\right\} = 0. \tag{38}$$

Now, our problem is to decide whether (35) is an extra condition to be enforced on $\mathbf{u}_0^*$ and χ_0 or whether it holds automatically. A moment's reflexion suggests that it should hold in some sense. In fact (36) is merely the result of subtracting the projections of the Euler equations on the two sides of the sheet onto the plane tangential to the sheet; on the other hand, everywhere between the sheets we have used irrotationality, conservation of mass and the component of the Euler equation normal to the sheet and this implies that the full Euler equation holds between the sheets. In order to render our argument a little more precise we take advantage of irrotationality to conclude that, for the exact solution, one has

$$[\mathbf{u}] = \nabla_T \Gamma, \tag{39}$$

where Γ is the jump in velocity potential across the sheet and ∇_T the gradient operator along it. The first approximation to (39) leads to

$$-2\pi \mathbf{n}_0 \wedge \boldsymbol{\omega}_0^* / |\nabla \chi_0| = \nabla_T \Gamma_0, \tag{40}$$

and to be sure that there exists a Γ_0 such that (40) holds we have to check that

$$\oint_{\mathscr{L}_0} \frac{\mathbf{n}_0 \wedge \boldsymbol{\omega}_0^*}{|\nabla \chi_0|} . \mathbf{ds} = 0 \tag{41}$$

for any closed contour $\mathscr{L}_0$ drawn on the sheet. As $|\nabla \chi_0|^{-1}$ is proportional to the (small) distance between two consecutive turns of the sheet, (41) is the flux of $\boldsymbol{\omega}_0^*$ across the curved surface of a small cylinder with generators normal to the sheet and with the area bounded by $\mathscr{L}_0$ as its cross-section. Now, from the fact that $\boldsymbol{\omega}_0^*$ is solenoidal, we see that (41) will hold if the flux of $\boldsymbol{\omega}_0^*$ across the ends of the cylinder is zero, but this is a consequence of (27). We conclude that (27) guarantees that (40) holds. Now, in order to check that (38) holds, we need check only that

$$\mathbf{n}_0 \wedge \{(\partial/\partial t + \mathbf{u}_0^* . \nabla) \nabla_T \Gamma_0 + (\nabla_T \Gamma_0 . \nabla) \mathbf{u}_0^*\} = 0, \tag{42}$$

or†

$$\mathbf{n}_0 \wedge \left\{ \nabla_T \left(\frac{\partial \Gamma_0}{\partial t} + \mathbf{u}_0^* . \nabla_T \Gamma_0 \right) - (\nabla \mathbf{u}_0^*) . \nabla_T \Gamma_0 + ((\nabla_T \Gamma_0) . \nabla) \mathbf{u}_0^* \right\} = 0, \tag{43}$$

and from (27) this holds if

$$\partial \Gamma_0 / \partial t + \mathbf{u}_0^* . \nabla_T \Gamma_0 = 0. \tag{44}$$

We said previously that (38) is satisfied automatically in some sense and we have just proved that this is true provided that (44) holds. Of course (44) is a well-known condition in the theory of vortex sheets and it is not at all surprising that we recover it. For the conical vortex sheet that we consider in the next section it is found that even (44) is satisfied automatically.

4. Application to the core of a leading-edge vortex

4.1. *General setting*

Let us apply the previous analysis to the problem of a conical rolled-up vortex sheet. We use spherical co-ordinates (r, θ, ϕ), the unit vectors being $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$, and set

$$\mathbf{u} = w\mathbf{e}_r + u\mathbf{e}_\theta + v\mathbf{e}_\phi \tag{45}$$

for the velocity. We look for a solution involving a conical vortex sheet highly

† The simplest way to justify the step from (42) to (43) is to continue Γ_0 outside the sheet. On the other hand, in $(\nabla \mathbf{u}_1^*) . \nabla_T \Gamma_0$ the indices of $\mathbf{u}_0^*$ are contracted with those of ∇_T.

rolled-up around $\theta = 0$. The first step is to obtain a solution of the inviscid incompressible flow equations with conical symmetry, i.e. a solution of

$$u_0^* \sin\theta \frac{\partial u_0^*}{\partial \theta} + v_0^* \frac{\partial u_0^*}{\partial \phi} + \frac{\partial p_0^*}{\partial \theta} + u_0^* w_0^* \sin\theta - v_0^{*2}\cos\theta = 0, \tag{46a}$$

$$u_0^* \sin\theta \frac{\partial v_0^*}{\partial \theta} + v_0^* \frac{\partial v_0^*}{\partial \phi} + \frac{\partial p_0^*}{\partial \phi} + v_0^*(u_0^* \cos\theta + w_0^* \sin\theta) = 0, \tag{46b}$$

$$u_0^* \sin\theta \frac{\partial w_0^*}{\partial \theta} + v_0 \frac{\partial w_0^*}{\partial \phi} - (u_0^{*2} + v_0^{*2})\sin\theta = 0, \tag{46c}$$

$$\frac{\partial (u_0^* \sin\theta)}{\partial \theta} + \frac{\partial v_0^*}{\partial \phi} + 2w_0^* \sin\theta = 0. \tag{46d}$$

For our problem we shall use the solution of (46) for θ small and we expect that the leading approximation will be independent of ϕ. It is a very easy matter to obtain a solution of (46) independent of ϕ and this has been worked out by Hall (1961). We characterize this particular solution by a further subscript zero. The following set of equations may easily be obtained:

$$\frac{d}{d\theta}\left(\frac{u_{00}^*}{v_{00}^{*2}\sin\theta}\right) = 0, \quad \text{giving} \quad C_0 u_{00}^* + v_{00}^{*2}\sin\theta = 0, \tag{47}$$

$$\frac{dw_{00}^*}{d\theta} - u_{00}^* + \frac{C_0}{\sin\theta} = 0, \tag{48}$$

$$\frac{du_{00}^*}{d\theta} + w_{00}^* + \frac{u_{00}^*\cos\theta + w_{00}^*\sin\theta}{\sin\theta} = 0, \tag{49}$$

where C_0 is an arbitrary constant. We integrate these equations to

$$u_{00}^* = -C_0\{(1-\cos\theta)\cot\theta + (-\log\tan\tfrac{1}{2}\theta + E_0)\sin\theta\}, \tag{50a}$$

$$v_{00}^* = C_0\{(-\log\tan\tfrac{1}{2}\theta + E_0) + (1-\cos\theta)\cos\theta\,(\operatorname{cosec}\theta)^2\}^{\frac{1}{2}}, \tag{50b}$$

$$w_{00}^* = C_0\{(-\log\tan\tfrac{1}{2}\theta + E_0)\cos\theta - (1-\cos\theta)\} \tag{50c}$$

(E_0 = constant). For small θ this leads to

$$\left.\begin{aligned} u_{00}^* &= -C_0\theta\Lambda + O(\theta^3|\log\theta|), \\ v_{00}^{*} &= C_0\Lambda^{\frac{1}{2}} + O(\theta^2|\log\theta|), \\ w_{00}^{*} &= C_0(\Lambda - \tfrac{1}{2}) + O(\theta^2|\log\theta|), \end{aligned}\right\} \tag{51}$$

with
$$\Lambda = -\log\tfrac{1}{2}\theta + E_0 + \tfrac{1}{2}, \tag{52}$$

and p_{00}^* is readily found to be

$$p_{00}^* = -\tfrac{1}{2}C_0^2\Lambda^2 + O(\theta^2|\log\theta|^2). \tag{53}$$

Having obtained a basic solution of (46) with axial symmetry, we can try to get more general solutions by perturbing it as follows:

$$u_0^*(\theta,\phi) = u_{00}^*(\theta) + \sum_{n\geqslant 1}\operatorname{Re}(u_{01,n}^*(\theta)\,e^{in\phi}) + \ldots, \tag{54}$$

with analogous formulae for v_0^*, w_0^* and p_0^*. Substituting (54) and the analogous formulae into (46) we get the ordinary differential equation

$$\mathbf{A}(\theta)\,\partial\mathscr{U}^*_{01,n}/\partial\theta+(in\mathbf{B}+\mathbf{C})\,\mathscr{U}^*_{01,n}=0 \tag{55}$$

for

$$\mathscr{U}^*_{01,n}=(u^*_{01,n},v^*_{01,n},w^*_{01,n},p^*_{01,n})^{\mathrm{T}}, \tag{56}$$

where the superscript T indicates the transpose of the row vector. It may be readily checked that the matrix **A** is not singular in the vicinity of $\theta=0$ except at $\theta=0$ itself, which is an irreguar singularity in the neighbourhood of which the structure of the solutions of (55) may be obtained through straightforward but tedious reductions. We shall not pursue this here.

4.2. *Further comments on the Mangler & Weber solution*

From now on, we restrict our attention to building up a solution exhibiting a rolled-up sheet using u^*_{00}, v^*_{00}, w^*_{00} and p^*_{00}. The first step is to find χ_0, which we write as χ_{00} in order to emphasize that it is based on the approximationof rotational symmetry for the model solution (u_0^*, v_0^*, w_0^*, p_0^*). We find

$$\chi_{00}=X\left(\int_\theta^{\Theta_{00}(\phi)}\frac{dt}{\epsilon_{00}(t)}\right),\quad \frac{d\Theta_{00}}{d\phi}=\epsilon_{00}[\Theta_{00}(\phi)], \tag{57}$$

where

$$\left.\begin{aligned}\epsilon_{00}(\theta)=u^*_{00}(\theta)\sin\theta/v^*_{00}(\theta)&=\theta^2\bar{\epsilon}_{00}(\theta)+O(\theta^2|\log\theta|),\\ \bar{\epsilon}_{00}(\theta)&=-\Lambda^{\frac{1}{2}}\end{aligned}\right\} \tag{58}$$

and where X stands for an arbitrary function. We now have to consider the vorticity $\boldsymbol{\omega}^*_{00}$ and prove that we can find a scalar function $\Gamma_{00}(r,\theta)$, defined on the sheet, such that (40) and (44) hold. We set

$$\boldsymbol{\omega}^*_{00}=\zeta^*_{00}\,\mathbf{e}_r+\xi^*_{00}\,\mathbf{e}_\theta+\eta^*_{00}\,\mathbf{e}_\phi \tag{59}$$

and compute that

$$\left.\begin{aligned}\xi^*_{00}&=-v^*_{00}\,r^{-1},\\ \eta^*_{00}=\left(u^*_{00}-\frac{dv^*_{00}}{d\theta}\right)r^{-1},&\quad \zeta^*_{00}=\frac{d(v^*_{00}\sin\theta)}{d\theta}(r\sin\theta)^{-1}.\end{aligned}\right\} \tag{60}$$

But

$$\nabla\chi_{00}=r^{-1}X'\{-\epsilon_{00}^{-1}\,\mathbf{e}_\theta+(\sin\theta)^{-1}\,\mathbf{e}_\phi\}, \tag{61}$$

and (27) is readily found to be equivalent to

$$v^{*2}_{00}+u^*_{00}(u^*_{00}-dw^*_{00}/d\theta)=0, \tag{62}$$

which holds for our model solution. From (40) we find that Γ_{00} must be such that†

$$\frac{\partial\Gamma_{00}}{\partial r}=-\frac{2\pi}{X'}\left(\frac{1}{\sin^2\theta}+\frac{1}{\epsilon^2_{00}}\right)^{-1}\left\{\frac{v^*_{00}}{\sin\theta}+\frac{1}{\epsilon_{00}}\left(\frac{dw^*_{00}}{d\theta}-u^*_{00}\right)\right\}, \tag{63a}$$

$$\frac{\partial\Gamma_{00}}{\partial\theta}=-\frac{2\pi}{X'}\frac{dv^*_{00}\sin\theta}{d\theta}, \tag{63b}$$

from which

$$\Gamma_{00}=(-2\pi/X')\,rv^*_{00}\sin\theta, \tag{64}$$

† We call the reader's attention to the fact that when $\nabla_{\mathrm{T}}\Gamma_{00}$ is computed, X' is constant.

provided that

$$\left(\frac{1}{\sin^2\theta}+\frac{1}{\epsilon_{00}^2}\right) v_{00}^* \sin\theta = \frac{v_{00}^*}{\sin\theta}+\frac{1}{\epsilon_{00}}\left(\frac{dw_{00}^*}{d\theta}-u_{00}^*\right), \tag{65}$$

which may readily be checked to be true.

Summing up, we have found that, provided that $r|\nabla\chi_{00}| \gg 1$, i.e. provided that

$$X'\{\epsilon_{00}^{-2}+(\sin\theta)^{-2}\}^{\frac{1}{2}} \gg 1, \tag{66}$$

we may build up a solution exhibiting a rolled-up sheet:

$$\mathbf{u} = (\mathbf{u}_{00}^*+\ldots)+(\overline{\mathbf{u}_{10}^*}+\ldots)+(\nabla\chi_{00}\wedge\boldsymbol{\omega}_{00}^*/|\nabla\chi_{00}|^2+\ldots)Y_1(\chi)+\ldots, \tag{67}$$

$$p = (p_{00}^*+\ldots)+\overline{p_{10}^*}+\ldots. \tag{68}$$

The closeness assumption may be satisfied in three ways: (i) $\theta \ll 1$, (ii) $E_0 \gg 1$ or (iii) $|X'| \gg 1$. The first two situations lead to a highly rolled vortex sheet while the third corresponds not to one sheet with closely spaced turns but to many sheets closely spaced from each other. It is not clear to what physical situation (iii) is appropriate. When (i) holds, slenderness and closeness are intimately related, while under (ii) they play separate roles. Mangler & Weber's solution is obtained from (57), (67) and (68) with $X' = 1$, when the model solution $\mathbf{u}_0^*$ is restricted to $\mathbf{u}_{00}^*$ and to $\theta \ll 1$ according to (51) and (53); then it may be shown that $\overline{u_{10}^*}$, $\overline{v_{10}^*}$, $\overline{w_{10}^*}$ and $\overline{p_{10}^*}$ are zero and that χ_{10} is zero. Under assumption (i) we expect the Mangler & Weber solution to be appropriate for the delta-wing leading-edge problem to leading order. Again, higher-order corrections to Mangler & Weber's solution of two kinds should be found. The first is a correction with respect to slenderness and should use solutions to (55) and then compute a slenderness correction χ_{01} to χ_{00}; this would bring in ellipticity for the vortex sheet. The second is a correction with respect to closeness and leads to

$$u = -C_0\theta\Lambda+\ldots+\{C_0\theta^2(\Lambda-\tfrac{1}{2})\Lambda^{\frac{1}{2}}+\ldots\}Y_1(\chi) + \{\overline{u_{20}^*}+\ldots-C_0\theta^3(5\Lambda^2-3\Lambda+\tfrac{1}{4})Y_2(\chi)\}+\ldots, \tag{69a}$$

$$v = C_0\Lambda^{\frac{1}{2}}+\ldots+\{-C_0\theta(\Lambda-\tfrac{1}{2})+\ldots\}Y_1(\chi) + \{\overline{v_{20}^*}+\ldots+C_0\theta^2\Lambda^{\frac{1}{2}}(2\Lambda-2)Y_2(\chi)\}+\ldots, \tag{69b}$$

$$w = C_0(\Lambda-\tfrac{1}{2})+\ldots+\{C_0\theta\Lambda^{\frac{1}{2}}+\ldots\}Y_1(\chi) + \{\overline{w_{20}^*}+\ldots-C_0\theta^2(\Lambda-\tfrac{1}{2})Y_2(\chi)\}+\ldots, \tag{69c}$$

$$p = -\tfrac{1}{2}C_0^2\Lambda^2+\ldots+\{\overline{p_{20}^*}-2C_0^2\theta^2(\Lambda^2-1)Y_2(\chi)+\ldots. \tag{69d}$$

Using the trick which allows u_{00}^*, v_{00}^* and w_{00}^* to be found, we get through straightforward but tedious calculations

$$\overline{u_{20}^*} = -\tfrac{1}{2}C_0\theta^2\{(8\pi^2-2)\Lambda-\tfrac{5}{2}+9\pi^2-12\pi^2\Lambda\mu-12\pi^2\gamma\}; \tag{70a}$$

$$\overline{v_{20}^*} = -\tfrac{1}{6}C_0\theta^2\{5\pi^2\Lambda^2-4\pi^2\Lambda+6\pi^2\Lambda\mu+3\pi^2\gamma+\tfrac{5}{8}-\tfrac{5}{2}\pi^2\}\Lambda^{-\frac{1}{2}}; \tag{70b}$$

$$\overline{w_{20}^*} = \tfrac{1}{2}C_0\theta^2\{(\tfrac{8}{3}\pi^2-1)\Lambda-2\pi^2\Lambda\mu+\tfrac{2}{3}(2\pi^2-1)\}; \tag{70c}$$

$$\overline{p_{20}^*} = -\tfrac{1}{6}C_0\theta^2\{4\pi^2\Lambda^2-6\pi^2\Lambda^2\mu-3\pi^2\Lambda\mu+(4\pi^2-\tfrac{1}{2})\Lambda-3\pi^2\gamma+\tfrac{3}{4}\pi^2+\tfrac{3}{8}\}, \tag{70d}$$

where

$$\mu(\theta) = -\frac{1}{D_0\theta^2}\int_0^{D_0\theta^2}\frac{dx}{\log x}, \quad \log D_0 = -(1+2E_0+2\log 2),$$

$$\gamma(\theta) = \tfrac{1}{2}\mu(\theta) + \frac{1}{2D_0^2\theta^4}\int_0^{D_0^2\theta^4}\frac{dx}{\log x}.$$

We may now, assuming slenderness, compute a closeness correction to the shape of the sheet as follows: as the sheet is given by $\chi = (2k+1)\pi$, we have, with $X' = 1$,

$$\chi = \int_\theta^{\Theta(\phi)}\frac{dt}{\epsilon(t)}, \quad \frac{d\Theta}{d\phi} = \epsilon[\Theta(\phi)], \tag{71}$$

$$\epsilon(\theta) = \theta^2\{\bar{\epsilon}_{00}(\theta)+\bar{\epsilon}_{20}(\theta)+\ldots\}, \tag{72}$$

$$\bar{\epsilon}_{00}(\theta) = -\Lambda^{\frac{1}{2}}, \tag{73a}$$

$$\bar{\epsilon}_{20}(\theta) = -\tfrac{1}{2}\theta^2\{\tfrac{11}{3}\pi^2\Lambda^2 - \pi^2\gamma - \tfrac{2}{3}(\pi^2+1)\Lambda - \tfrac{5}{24} + \tfrac{5}{6}\pi^2\}\Lambda^{-\frac{1}{2}}, \tag{73b}$$

which would allow computation of χ_{20}, the leading approximation to χ_2 with respect to slenderness.

5. Application to Kaden's problem

5.1. *Summary of Moore's results*

As a second application of the present theory we consider Kaden's problem, which was worked out by Moore (1975), and show that complete agreement is obtained with his results. At times $t < 0$ we consider two-dimensional, incompressible, steady, irrotational flow around an edge, with complex velocity potential

$$\phi + i\psi = -\gamma(x+iy)^{\frac{1}{2}}. \tag{74}$$

From $t = 0$ on, we remove the edge. It is found that a vortex sheet begins to roll up. From dimensional considerations, the focus of the spiral is expected to be at $x = a(\gamma t)^{\frac{2}{3}}$, $y = b(\gamma t)^{\frac{2}{3}}$, where a and b are numerical constants, and accordingly we set

$$x = a(\gamma t)^{\frac{2}{3}} + r\cos\theta, \quad y = b(\gamma t)^{\frac{2}{3}} + r\sin\theta. \tag{75}$$

Again from dimensional considerations, as shown by Kaden (1931), the shape of the sheet is expected to be given by

$$\theta = \epsilon - \beta\xi^{-1}\{1+\mathscr{F}(\xi,\theta)\}, \tag{76}$$

where

$$\xi = r^{\frac{3}{2}}/\gamma t. \tag{77}$$

The focus of the spiral is at $\xi = 0$, which corresponds either to $r \to 0$, t fixed or to $t \to \infty$, r fixed. From Moore's solution

$$\beta = (2\pi\alpha^{\frac{1}{2}})^{-1}, \tag{78}$$

where α, as well as ϵ, is a numerical constant. The function $\mathscr{F}$ may be expanded when $\xi \to 0$, and Moore derived the leading term of this expansion:

$$\mathscr{F} = \tfrac{3}{2}C(2\pi\alpha^{\frac{1}{2}})^n\xi^n\cos[2(\theta-\theta_0)+\phi]+\ldots, \tag{79a}$$

$$C = |A|+|B|, \quad A = |A|e^{i\phi}, \quad B = |B|e^{-i\phi}, \tag{79b–d}$$

$$9|A|+\frac{3(1+n)}{\frac{5}{3}+n}|B| = 0, \quad 3|B|-\frac{1-3n}{1-n}|A| = 0. \tag{79e}$$

Three further terms will be derived below. By a dimensional argument the strength of the sheet is expected to be given by

$$\Gamma \equiv [\phi] = \lambda\gamma\alpha^{-\frac{1}{2}}r^{\frac{1}{2}}\{1+G(\xi,\theta)\}, \tag{80}$$

where λ is some numerical constant. From Moore's solution we find that

$$\lambda = 1, \tag{81}$$

and that the leading term in the expansion of G reads

$$G = -\tfrac{1}{2}C(2\pi\alpha^{\frac{1}{2}})^n\xi^n\cos[2(\theta-\theta_0)+\phi]+\ldots. \tag{82}$$

This result will be rederived below and we shall give three further terms in the expansion of G. We observe that there are three adjustable constants in Moore's solution, namely α, ϵ and C. These constants cannot be determined from local considerations only and they are related to the overall shape of the sheet. Two other arbitrary constants will be added by the three further terms computed below.

5.2. *The closeness expansion*

Setting the origin at $x = a(\gamma t)^{\frac{2}{3}}$, $y = b(\gamma t)^{\frac{2}{3}}$ and using the polar co-ordinates introduced in (75), we write

$$\mathbf{u} = u\mathbf{e}_r+v\mathbf{e}_\theta, \tag{83}$$

for the velocity vector, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ standing for unit vectors in the radial and azimuthal directions respectively.

The zeroth-order approximation with respect to closeness is an unsteady, incompressible, two-dimensional, rotational flow, with time entering only through ξ, as defined in (77). For such a flow we set

$$\left.\begin{aligned} u_0^* &= \gamma\{2\pi(\alpha r)^{\frac{1}{2}}\}^{-1}\mathscr{U}(\xi,\theta), \\ v_0^* &= -\gamma\{2\pi(\alpha r)^{\frac{1}{2}}\}^{-1}\{1+\mathscr{V}(\xi,\theta)\}. \end{aligned}\right\} \tag{84}$$

From Kaden's solution and Moore's improvement on it, we expect that $\mathscr{U}$ and $\mathscr{V}$ go to zero with ξ. The vorticity is easily computed to be $\omega_0^*\,\mathbf{k}$, where $\mathbf{k}$ is a unit vector normal to the plane of flow and

$$\omega_0^* = -\tfrac{1}{2}\gamma\{2\pi(\alpha r)^{\frac{1}{2}}\}^{-1}r^{-1}\{1+\Omega(\xi,\theta)\}, \tag{85}$$

with

$$\Omega = (1+3\xi\partial/\partial\xi)\mathscr{V}+2\partial\mathscr{U}/\partial\theta. \tag{86}$$

From the continuity and vorticity equations we get

$$\mathscr{U}+3\xi\partial\mathscr{U}/\partial\xi-2\partial\mathscr{V}/\partial\theta = 0, \tag{87}$$

$$\frac{\partial\Omega}{\partial\theta}+\tfrac{3}{2}\mathscr{U}+\mathscr{V}\frac{\partial\Omega}{\partial\theta}+\tfrac{3}{2}\mathscr{U}\left(1-\xi\frac{\partial}{\partial\xi}\right)\Omega+2\pi\alpha^{\frac{1}{2}}\xi^2\frac{\partial\Omega}{\partial\xi} = 0, \tag{88}$$

and from (86)–(88) we find

$$\left[\left(1+3\xi\frac{\partial}{\partial\xi}\right)^2+3\right]\mathscr{U}+4\frac{\partial^2\mathscr{U}}{\partial\theta^2} = -\left\{2\mathscr{V}\frac{\partial\Omega}{\partial\theta}+3\mathscr{U}\left(1-\xi\frac{\partial}{\partial\xi}\right)\Omega+4\pi\alpha^{\frac{1}{2}}\xi^2\frac{\partial\Omega}{\partial\xi}\right\}, \tag{89}$$

which will prove useful later on when we look for a slenderness expansion of $\mathscr{U}$, $\mathscr{V}$ and Ω. Here we consider only the limitation imposed on the possible solutions by periodicity with respect to θ. We define $\bar{f}$ to be the average of the periodic function f according to

$$\bar{f} = \frac{1}{2\pi}\int_0^{2\pi} f\,d\theta. \tag{90}$$

Averaging (87) and taking into account the fact that $\mathscr{U}$ is bounded (indeed vanishing) when ξ goes to zero, we find

$$\overline{\mathscr{U}} = 0; \tag{91}$$

then on averaging (86) amd (88) we get

$$\overline{\Omega} = (1+3\xi\partial/\partial\xi)\overline{\mathscr{V}}, \tag{92}$$

$$\overline{2\mathscr{V}\frac{\partial\Omega}{\partial\theta}}+\overline{3\mathscr{U}\left(1-\xi\frac{\partial}{\partial\xi}\right)\Omega}+4\pi\alpha^{\frac{1}{2}}\xi^2\frac{\partial\overline{\Omega}}{\partial\xi} = 0. \tag{93}$$

When dealing later with the slenderness expansion of the solution of (86)–(88), we shall find (91)–(93) very useful in that they will allow determination of some constants which otherwise could be determined only by continuing rather far with the expansion process.

From now on, we assume that we know a solution of (86)–(88) such that $\mathscr{U}$, $\mathscr{V}$ and Ω go to zero when $\xi \to 0$ and such that (91)–(93) hold. We look now for the first correction to that solution with respect to closeness. The first step is to find a χ_0 such that (21) holds and such that the equation of the sheet is $\chi_0 = \pi$ to leading order with respect to closeness. Guided by Moore's solution, we set

$$\chi_0 = \theta-\epsilon+\beta\xi^{-1}\{1+\mathscr{F}(\xi,\theta)\}, \tag{94}$$

and find that (21) leads to

$$\begin{aligned}(1-2\pi\alpha^{\frac{1}{2}}\beta)\xi+\xi\{\mathscr{V}-2\pi\alpha^{\frac{1}{2}}\beta(1-\xi\partial/\partial\xi)\mathscr{F}\}&\\ +\beta\{\tfrac{3}{2}\mathscr{U}[1+(1-\xi\partial/\partial\xi)\mathscr{F}]+(1+\mathscr{V})\,\partial\mathscr{F}/\partial\theta\} &= 0.\end{aligned} \tag{95}$$

On averaging we find

$$(1-2\pi\alpha^{\frac{1}{2}}\beta)\xi+\xi\left\{\overline{\mathscr{V}}-2\pi\alpha^{\frac{1}{2}}\beta\left(1-\xi\frac{\partial}{\partial\xi}\right)\overline{\mathscr{F}}\right\}+\beta\left\{\tfrac{3}{2}\overline{\mathscr{U}\mathscr{F}}-\tfrac{3}{2}\xi\overline{\mathscr{U}\frac{\partial\mathscr{F}}{\partial\xi}}+\overline{\mathscr{V}\frac{\partial\mathscr{F}}{\partial\theta}}\right\} = 0. \tag{96}$$

Now, assuming that we know a solution of (95) such that (96) holds, we find first that

$$|\nabla\chi_0|^2 = \tfrac{9}{4}(\xi r)^{-2}\{\beta^2(1+\mathscr{F}-\xi\,\partial\mathscr{F}/\partial\xi)^2+\tfrac{4}{9}(\beta d\mathscr{F}/d\theta+\xi)^2\}, \tag{97}$$

then, from (18), (24), (84), (94) and (97), find that the first two terms of the closeness expansion read

$$u^* = u_0^*-\frac{[u_1^*]}{2\pi}Y_1(\chi), \quad v^* = v_0^*-\frac{[v_1^*]}{2\pi}Y_1(\chi), \tag{98}$$

where

$$\left.\begin{aligned}[u_1^*] &= \tfrac{2}{9}\gamma\alpha^{-\frac{1}{2}}r^{-\frac{1}{2}}\xi\,\frac{(\beta\,\partial\mathscr{F}/\partial\theta+\xi)\,(1+\Omega)}{\beta^2(1+\mathscr{F}-\xi\partial\mathscr{F}/\partial\xi)^2+\frac{4}{9}(\beta\,\partial\mathscr{F}/\partial\theta+\xi)^2},\\ [v_1^*] &= \tfrac{1}{3}\gamma\alpha^{-\frac{1}{2}}r^{-\frac{1}{2}}\xi\,\frac{(1+\mathscr{F}-\xi\,\partial\mathscr{F}/\partial\xi)\,(1+\Omega)}{\beta^2(1+\mathscr{F}-\xi\,\partial\mathscr{F}/\partial\xi)^2+\frac{4}{9}(\beta\,\partial\mathscr{F}/\partial\theta+\xi)^2}.\end{aligned}\right\} \tag{99}$$

From the obvious relation

$$|[v_1^*]|/|v_0^*| = O(\xi), \tag{100}$$

we see that the closeness parameter is equal to ξ. Thus we cannot expect (98) to be useful unless ξ is small but then it will be sufficient to have approximations of $\mathscr{U}$, $\mathscr{V}$, Ω and $\mathscr{F}$ for small ξ. Let us observe that, according to (28) and (29), $\overline{u_1^*}$ and $\overline{v_1^*}$ may be considered as part of the slenderness expansion of u_0^* and v_0^*. We shall accordingly set

$$\overline{u_1^*} = \overline{v_1^*} = 0. \tag{101}$$

In order to construct the first approximation in closeness we need now to find the strength Γ_0 of the sheet, according to (40). Guided by Moore's result (80) we set

$$\Gamma_0 = \lambda\gamma(\alpha/r)^{-\frac{1}{2}}\{1+G(\xi,\theta)\}, \tag{102}$$

and get for G the equation

$$(1-\lambda)\xi-\lambda\beta\left\{\frac{\partial\mathscr{F}}{\partial\theta}\left[1+\left(1+3\xi\frac{\partial}{\partial\xi}\right)G\right]\right.$$
$$\left.+3\frac{\partial G}{\partial\theta}\left[1+\left(1-3\xi\frac{\partial}{\partial\xi}\right)\mathscr{F}\right]\right\}+\xi\Omega-\lambda\xi\left(1+3\xi\frac{\partial}{\partial\xi}\right)G = 0. \tag{103}$$

Averaging this equation, we find

$$(1-\lambda)\xi-\lambda\beta\left\{3\xi\overline{\frac{\partial G}{\partial\xi}\frac{\partial\mathscr{F}}{\partial\theta}}+\overline{G\frac{\partial\mathscr{F}}{\partial\theta}}-3\xi\overline{\frac{\partial\mathscr{F}}{\partial\xi}\frac{\partial G}{\partial\theta}}\right.$$
$$\left.+3\overline{\mathscr{F}\frac{\partial G}{\partial\theta}}\right\}+\xi\overline{\Omega}-\lambda\xi\left(1+3\xi\frac{\partial}{\partial\xi}\right)\overline{G} = 0. \tag{104}$$

Now, provided that we know a solution of (103) such that (104) holds, we have constructed the first two terms of the closeness expansion and we stop the process here.

5.3. *The slenderness expansion*

We come now to the process of constructing an approximate solution of (86)–(88) when ξ is small. We try

$$\left.\begin{aligned}\mathscr{U} &= \sum_{k=1}^{K}\xi^{n_k}\mathscr{U}_k(\theta)+o(\xi^{n_K}),\\ \mathscr{V} &= \sum_{k=1}^{K}\xi^{n_k}\mathscr{V}_k(\theta)+o(\xi^{n_K}),\\ \Omega &= \sum_{k=1}^{K}\xi^{n_k}\Omega_k(\theta)+o(\xi^{n_K}).\end{aligned}\right\} \tag{105}$$

k	1	2	3	4	5	6	7	8	9	10	11
n_k	ν_2	ν_3	$2\nu_2$	ν_2+1	ν_4	$\nu_2+\nu_3$	ν_3+1	$3\nu_2$	$2\nu_2+1$	ν_2+2	ν_5
	0·8685	1·5815	1·7370	1·8685	2·2701	2·4500	2·5815	2·6055	2·7370	2·8685	2·9496

TABLE 1

The first step is to find the sequence $\{n_k\}$. Substitution of (105) into (86)–(88) leads to a system of recurrence relations. At each step k one has to solve a linear system for $\mathscr{U}_k$, $\mathscr{V}_k$ and Ω_k with a forcing term involving $\mathscr{U}_J$, $\mathscr{V}_J$ and Ω_J with $J < k$. Then it is easily found by eliminating $\mathscr{V}_k$ and Ω_k that

$$[(1+3n_k)^2+3]\,\mathscr{U}_k+4\,d^2\mathscr{U}_k/d\theta^2 = (\text{R.H.S.})_k, \tag{106}$$

where $(\text{R.H.S.})_k$ involves $\mathscr{U}_J$, $\mathscr{V}_J$ and Ω_J and their derivatives for $J < k$ only, a point which may be seen from (89). Looking for a solution of the homogeneous version of (106) of the form $\mathscr{U}_k = a_k \sin[q(\theta+\phi_k)]$ we find that the sequence of n_k includes all ν_q such that†

$$\nu_q = \tfrac{1}{3}\{(4q^2-3)^{\frac{1}{2}}-1\}, \quad q \geqslant 2, \quad q\,\text{integer}. \tag{107}$$

Furthermore the very process of building the solution shows that the sequence $\{n_k\}$ must be stable to the addition of unity to any member of the sequence as well as to the addition of any two members of the sequence. This will work as long as neither of these processes leads to some ν_q. When this occurs it will be found necessary to introduce into the expansion (105) logarithmic terms such as $\xi^{n_k}(\log\xi)^p$, where $p \leqslant \mathscr{P}(k)$, p being an integer. We have not found to what order this will occur if at all, but table 1 shows that this occurrence is not expected in any practical application of the expansion.

Using (86)–(88) and (91)–(93), we compute the first four terms of each expansion:

$$\left.\begin{aligned} &\mathscr{U}_1 = a_1\cos 2\psi_1, \qquad \mathscr{U}_2 = a_2\cos 3\psi_2,\\ &\mathscr{U}_3 = -\frac{a_1^2}{8(n_1+1)}\sin 4\psi_1, \quad \mathscr{U}_4 = \frac{\pi\alpha^{\frac{1}{2}}n_1}{6n_1+5}a_1\sin 2\psi_1,\\ \text{where}\quad &a_1 = -2C(2\pi\alpha^{\frac{1}{2}})^{n_1}; \end{aligned}\right\} \tag{108}$$

$$\left.\begin{aligned} &\mathscr{V}_1 = \tfrac{1}{4}(1+3n_1)\,a_1\sin 2\psi_1, \qquad \mathscr{V}_2 = \tfrac{1}{6}(1+3n_2)\,a_2\sin 3\psi_2,\\ &\mathscr{V}_3 = \frac{1}{64}\frac{1+6n_1}{n_1+1}a_1^2\cos 4\psi_1+\overline{\mathscr{V}_3}, \quad \mathscr{V}_4 = -\frac{\pi\alpha^{\frac{1}{2}}}{2}\frac{n_1+2}{6n_1+5}\,a_1\cos 2\psi_1, \end{aligned}\right\} \tag{109}$$

$$\left.\begin{aligned} &\Omega_1 = -\tfrac{3}{4}a_1\sin 2\psi_1, \qquad \Omega_2 = -\tfrac{1}{2}a_2\sin 3\psi_2,\\ &\Omega_3 = \frac{3}{64}\frac{4n_1+5}{n_1+1}a_1^2\cos 4\psi_1+\overline{\Omega_3}, \quad \Omega_4 = -\frac{6\pi\alpha^{\frac{1}{2}}}{6n_1+5}a_1\cos 2\psi_1, \end{aligned}\right\} \tag{110}$$

† For $q = 1$ we find $n_k = 0$ and this corresponds nearly to a shift in the location of the focus of the spiral as indicated by Moore (1975, §5 after formula (5.1)). We eliminate this term.

with $\psi_1 = \theta + \phi_1$ and $\psi_2 = \theta + \phi_2$, where a_1 and a_2 are arbitrary constants and ϕ_1 and ϕ_2 are arbitrary phase shifts; furthermore we have set

$$\overline{\mathscr{V}_3} = -\frac{3}{64} a_1^2, \quad \overline{\Omega_3} = -\frac{3}{64} \frac{45n_1+16}{n_1+8} a_1^2. \tag{111}$$

From (95) we find that the expansion of $\mathscr{F}$ reads

$$\mathscr{F} = \sum_{k=1}^{K} \xi^{n_k} \mathscr{F}_k + \sum_{J=1}^{J^*} \xi^J \tilde{\mathscr{F}}_J + \dots, \tag{112}$$

where the J are integers. Reordering we get

$$\mathscr{F} = \xi^{\nu_2} \mathscr{F}_1 + \xi \tilde{\mathscr{F}}_1 + \xi^{\nu_3} \mathscr{F}_2 + \xi^{2\nu_2} \mathscr{F}_3 + \xi^{\nu_2+1} \mathscr{F}_4 + \xi^2 \tilde{\mathscr{F}}_2 + \dots. \tag{113}$$

Referring to the general theory we see that the terms $\xi\tilde{\mathscr{F}}_1$ and $\xi^2\tilde{\mathscr{F}}_2$ may be counted in either of the two kinds of expansion. We observe that in principle it is not legitimate to include the term $\xi^2\tilde{\mathscr{F}}_2$ as long as the closeness expansion has been stopped just before terms $O(\xi^2)$. From (95) and (96) we find

$$\left.\begin{aligned} &\mathscr{F}_1 = -\tfrac{3}{4} a_1 \sin 2\psi_1, \quad \tilde{\mathscr{F}}_1 \equiv 0, \quad \mathscr{F}_2 = -\tfrac{1}{2} a_2 \sin 3\psi_2, \\ &\mathscr{F}_3 = -\frac{3}{64} \frac{4n_1+5}{n_1+1} a_1^2 \cos 4\psi_1 + \overline{\mathscr{F}}_3, \quad \mathscr{F}_4 = \frac{\pi\alpha^{\frac{1}{2}}}{4} \frac{27n_1+20}{6n_1+5} a_1 \cos 2\psi_1, \end{aligned}\right\} \tag{114}$$

where we have set

$$\overline{\mathscr{F}}_3 = \frac{1}{64} \frac{114n_1+439}{11-4n_1} a_1^2. \tag{115}$$

A word of explanation may be useful about the result $\tilde{\mathscr{F}}_1 \equiv 0$. This is readily obtained from (96): dividing by ξ and letting ξ go to zero in the resulting equation gives

$$1 - 2\pi\alpha^{\frac{1}{2}}\beta = 0, \tag{116}$$

then from (95) and (96) we obtain $\partial\tilde{\mathscr{F}}_1/\partial\theta = 0$ and $\overline{\tilde{\mathscr{F}}_1} = 0$.

In much the same way, from (103) and (104) we find the expansion of G to be

$$G = \xi^{\nu_2} G_1 + \xi^{\nu_3} G_2 + \xi^{2\nu_2} G_3 + \xi^{\nu_2+1} G_4 + \xi^2 \tilde{G}_2 + \dots, \tag{117}$$

with

$$\left.\begin{aligned} &G_1 = \tfrac{1}{4} a_1 \sin 2\psi_1, \quad G_2 = \tfrac{1}{6} a_2 \sin 3\psi_2, \\ &G_3 = \frac{1}{64} \frac{1}{n_1+1} a_1^2 \cos 4\psi_1 + \overline{G}_3, \quad G_4 = \frac{2\pi\alpha^{\frac{1}{2}}}{6n_1+5} a_1 \cos 2\psi_1 \end{aligned}\right\} \tag{118}$$

and

$$\overline{G}_3 = \frac{1}{64} \frac{2781n_1+3328}{141n_1+440} a_1^2. \tag{119}$$

5.4. *Comparison with Moore's results: comments*

From the values $\mathscr{F}_1$ in (114) and G_1 in (118), by referring to § 5.1 we see that there is complete agreement with Moore's results. These have been improved in two ways: first, we have added three more terms to Moore's expansions; second, we have given a consistent approximation for the flow variables all the way from one side of the sheet to the other. We think that with a moment's reflexion the reader will be convinced that our process is fairly simple. In fact the rolling-up process affects the closeness expansion only and, when limited to its first two terms, may be fairly simply obtained from the rotational solution. From the knowledge of $\mathscr{U}$, $\mathscr{V}$ and Ω, this calculation has nothing to do with the rolling-up process itself, and we find that all the computations are contained between (94) and the end of § 5.2. Of course, it is through the process of computing $\mathscr{U}$, $\mathscr{V}$ and Ω by the slenderness expansion that the computations become tedious. By the end we observe that Moore states that the error term in his expansions of $\mathscr{F}$ is $O(\xi)$; we have computed this and found it to be zero.

6. Discussion

We must recognize that the analysis of § 3 is a formal one, but we cannot expect to be able to render it more rigorous and we are led to make a conjecture which is strongly suggested. Suppose that we know a solution of (20) and (21) such that (27) holds, then, as we have shown, there exists a Γ_0 such that (40) holds too. Our conjecture is that, if $\chi_0 = (2k+1)\pi$ has the structure of a rolled-up sheet with $\chi_0 = \pm\pi$ very close to each other, then our solution $(\mathbf{u}_0^*, p_0^*)$ will be an approximate simulation of the core of a rolled-up vortex sheet. The leading-order vorticity of this sheet will be $-2\pi|\nabla\chi_0|^{-1}\boldsymbol{\omega}_0^*$ and appears in (24) and (25), which exhibit the cancellation of the distributed vorticity in the leading approximation by the vorticity of the next approximation, in such a way that the flow given by the first two approximations (considered together) is irrotational with, now, the vorticity fully concentrated on the sheet. A second approximation is contained in (32) and (33) and it appears that the vorticity associated with its derivatives with respect to χ cancels the vorticity associated with derivatives with respect to $\mathbf{x}$ coming from the first approximation, in such a way that, again, the first three approximations (considered together) are irrotational with vorticity concentrated on the sheet. The coefficients of $Y_1(\chi)$ in (24) and (33) are related to the vorticity concentrated on the sheet according to leading and first approximation respectively. There is no concentrated vorticity corresponding to the coefficient of $Y_2(\chi)$ in (33) and the occurrence of this term comes from the need to cancel the distributed vorticity associated with the variation with χ of the distributed vorticity arising from the coefficient of $Y_1(\chi)$ in (24).

Assume that $|\mathbf{u}_0^*|$ is of order one and that $\partial\mathbf{u}_0^*/\partial t$ and $|\nabla\mathbf{u}_0^*|$ are of order s^{-1}, where s is a slenderness parameter. Furthermore, assume that $\partial\chi_0/\partial t$ and $|\nabla\chi_0|$ are of order $c^{-1}s^{-1}$. Then from inspection of (24) we see that $\tilde{\mathbf{u}}_1^*$ is of order c while from inspection of (33) we see that $\tilde{\mathbf{u}}_2^*$ is the sum of two kinds of terms. The first ones are of order $|\overline{\mathbf{u}_1^*}|\,|\nabla\chi_0|^{-1}$, that is $s^{-1}|\overline{\mathbf{u}_1^*}|c$, while the others are of order c^2. Setting $\overline{\mathbf{u}_1^*}$, $\overline{p_1^*}$ and χ_1 all equal to zero we find that the successive terms in the expansions (18) are of orders c^0, c^1, c^2, ..., so that these expansions may be called closeness expansions. From (98) we see that the closeness parameter is a measure of the ratio of the discontinuity in velocity across the sheet to the mean of the velocities on either side of the sheet. Now we discuss the values of $\overline{\mathbf{u}_1^*}$, $\overline{p_1^*}$ and χ_1. From (28)–(31), they satisfy homogeneous equations, and setting $\overline{\mathbf{u}_1^*} = 0$, $\overline{p_1^*} = 0$ and $\chi_1 = 0$ gives a particular solution. Whether this is the correct one is not a question relevant to the closeness expansion alone. As a matter of fact the smallness of c is tied to $\mathbf{u}_0^*$ and χ_0 and is indeed a property of the rotational solution $\mathbf{u}_0^*$, at least if we deal with one tightly wound sheet (refer to the discussion after (68) for another interpretation). In the two applications that have been worked out, the smallness of c is related to the smallness of the slenderness parameter s and it is useful to expand $\mathbf{u}_0^*$, p_0^* and χ_0 with respect to slenderness if one wants to obtain an explicit solution for the basic flow. Then $\overline{\mathbf{u}_1^*}$, $\overline{p_1^*}$ and χ_1 are best considered as part of the slenderness expansion of $\mathbf{u}_0^*$, p_0^* and χ_0, thus it is legitimate to set them equal to zero. We have not worked out the equations for $\overline{\mathbf{u}_2^*}$, $\overline{p_2^*}$ and χ_2 but it is easily shown that they have to satisfy inhomogeneous equations and will be non-zero [that this is actually so may be checked from (70)]. It may be further checked that they are of order c^2 with respect to $\mathbf{u}_0^*$, p_0^* and χ_0, at least for the particular solution which is forced by the inhomogeneous terms, the complementary solution pertaining to the slenderness expansion.

Now let us comment on the determinacy or indeterminacy of the solution. In an earlier version of the paper we worked out only the application to the Mangler & Weber problem and one referee was led to suspect the validity of our procedure for the reason that there is no adjustable constant in (69). As a matter of fact this has to do with closeness and slenderness. Through inspection of (24) and (33) for example, we see that the closeness expansion up to the order which has been considered is fully determinate when $\mathbf{u}_0^*$, p_0^* and χ_0 are known. This amounts to saying that the irrotational solution with a rolled-up sheet is fully determined by the associated rotational solution with no sheet. This statement looks reasonable. If adjustable constants are necessary, for example in order to account for the observed ellipticity of the sheet, these must not be looked for in a modification of the theory which relates one solution to the other, and which is the essence of the closeness expansion; rather the adjustable constants have to be sought in a modification of the basic rotational solution ($\mathbf{u}_0^*$, p_0^*, χ_0). Unfortunately, with the leading-edge core it is very difficult to obtain such a modification as we have shown by the end of § 4.1, but with the Kaden's problem we have been able to achieve this goal and our solution contains probably more adjustable constants than it would prove practical to use in any check with numerical computations.

The most fruitful continuation of the present work would be to use it in order to devise an efficient numerical process for computation of complex flow configurations involving rolled-up vortex sheets as in Rehbach (1975). Another extension would be to investigate the use, as the basic rotational flow, of a solution involving slender vortex filaments according to the work of Ting (1971), Widnall, Bliss & Zalay (1971) and Moore & Saffman (1972).

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Note added in proof. The two points raised respectively at the very end of §§4 and 6 have been examined and are discussed in a paper to appear in the proceedings of the Journées Mathématiques sur les perturbations singulières et al théorie de la couche limité, Lyon 8–10 Decembre 1976, to be published in *Lecture Notes in Mathematics*, Springer Verlag. It is shown first that (44) holds automatically and second that a vortex sheet may roll around any slender vortex filament. If ϵ is the slenderness of the filament (diameter of its cross section over radius of the filament) the closeness is $C = \epsilon^2$ and it is found that there is a small ellipticity of $O(\epsilon)$.